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# INJECTIVITY OF GLOBAL MAPS OF CELLULAR AUTOMATA (Algebras, logics, languages and related areas)

AUTHOR(S):

Ishibashi, Hiroyuki

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# INJECTIVITY OF GLOBAL MAPS OF CELLULAR AUTOMATA

HIROYUKI ISHIBASHI

ABSTRACT. The purpose is to show briefly and visually that cellular automata  $\mathcal{A}$ 's of finite type with a quiescent state  $q$  are injective if and only if either  $\mathcal{A}$  contains two mutually erasable configurations  $c_1, c_2$  in Moore [2] or two not distinguished configurations  $d_1, d_2$  in Myhill [3].

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## 1. PRELIMINARIES

$\circ \mathcal{A} = \{\mathbb{Z}^2, S, N, f\}$  : a cellular automaton

where

$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$  : the rational integers,

$S = \{s_1, s_2, \dots, s_t\}$  : the state set ,

$$N(i) = \left\{ \begin{array}{ccc} i + (-1, 1) & i + (0, 1) & i + (1, 1) \\ i + (-1, 0) & i + (0, 0) & i + (1, 0) \\ i + (-1, -1) & i + (0, -1) & i + (1, -1) \end{array} \right\} \subseteq \mathbb{Z}^2$$

: neighbourhood of  $i$  for  $i = (i_1, i_2) \in \mathbb{Z}^2$  ,

so  $|N(i)| = 9$ ,

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- $f = \overbrace{S \times S \times \cdots \times S}^9 \longrightarrow S$  : local map ,
- $C = S^{\mathbb{Z}^2}$  : configuration set,
- $C \ni c = (\cdots, c(i), \cdots) : \mathbb{Z}^2 \rightarrow S$ , a map
- $F : C \rightarrow C$  : global map
- $c = (\cdots, c(i), \cdots) \mapsto c' = (\cdots, c'(i), \cdots)$   
 where  $c'(i) = f(c(N(i)))$  with  $c(N(i)) = c|_{N(i)}$
- $C_n = \{c|_{\Delta_n} \mid c \in C\}$   
 where  $\Delta_n$  is an  $n \times n$  - square subset of  $\mathbb{Z}^2$   
 (Note : Since  $F$  is homogeneous, the choice of  $\Delta_n$  in  $\mathbb{Z}^2$  is not essential. )
- $C_{n+2} = \{c|_{\Delta_{n+2}} \mid c \in C\}$   
 where  $\Delta_{n+2}$  is the extension of  $\Delta_n$  one cell on four sides.
- $C_{n+2} \setminus C_n = \{c|_{\Delta_{n+2} \setminus \Delta_n} \mid c \in C\}$   
 : the set called edges or frames
- $E = c \setminus c_n$  : environment of  $c_n$  ,  
 where  $c \in C$  and  $c_n \in C_n$  and we write  
 $c = c_n \vee E$

◦ For  $c = c_n \vee e$  with  $c \in C_{n+2}$ ,  $c_n \in C_n$  and  $e \in C_{n+2} \setminus C_n$ , we have

$$c(N(i)) \subseteq c \text{ for } i \in \Delta_n$$

This allows us to define a function

$$F_{n,e} : C_n \longrightarrow C_n$$

$$c_n = (\cdots, c_n(i), \cdots) \longmapsto c'_n = (\cdots, c'_n(i), \cdots)$$

with

$$c'_n(i) = f(c(N(i))) \text{ and } c = c_n \vee e$$

- $q \in S$  with  $f(\underbrace{q, q, \cdots, q}_9) = q$  : quiescent state.

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$$\circ c \in C \text{ with } |\{i \in \mathbb{Z}^2 \mid c(i) \neq q\}| < \infty$$

: a configuration of finite type,

$$\text{i.e., } c : \text{finite type} \Leftrightarrow c = c_n \vee E$$

for some  $c_n$  in  $C_n$  and  $E$  an environment of which states are all  $q$ .

♣ Assumption.

$$(1) \exists q \in S.$$

$$(2) \forall c \in C \Rightarrow c : \text{finite type}.$$

## 2. STATEMENT OF THE THEOREM

**Definition.**

$d_1, d_2 \in C_{n-4}$  **with**  $d_1 \neq d_2$  : **n-mutually erasable**

$$\Leftrightarrow \exists d' \in C_{n-4}, \exists g, g' \in C_{n-2} \setminus C_{n-4} \text{ and } \exists h \in C_n \setminus C_{n-2}$$

such that

$$\begin{array}{ccc} C_{n-2} \ni d_1 \vee g & \xrightarrow{F_{n-2,h}} & \\ & & d' \vee g' \in C_{n-2}. \\ C_{n-2} \ni d_2 \vee g & \xrightarrow{F_{n-2,h}} & \end{array}$$

**Remark.** (a) Let  $d_1, d_2$  be  $n$ -mutually erasable. Then, for any  $l$  in  $C_{n+2} \setminus C_n$  there exists  $h'$  in  $C_n \setminus C_{n-2}$  such that

$$\begin{array}{ccc} C_n \ni c_1 = d_1 \vee g \vee h & \xrightarrow{F_{n,l}} & \\ & & d' \vee g' \vee h' \in C_n. \\ C_n \ni c_2 = d_2 \vee g \vee h & \xrightarrow{F_{n,l}} & \end{array}$$

(b) Note that  $d'_1 = d_1 \vee g$  and  $d'_2 = d_2 \vee g$  are also  $(n+2)$ -mutually erasable by taking  $h, l$  for  $g, h$  and thus this procedure can be continued until to get their extensions  $\tilde{c}_1, \tilde{c}_2$  in  $C$ .

**Definition.**

$d_1, d_2 \in C_{n-4}$  **with**  $d_1 \neq d_2$  :  $n$ -**not distinguished**

$$\Leftrightarrow \exists c' \in C \text{ and } \exists E \in C \setminus C_{n-4}$$

such that

$$\begin{array}{ccc} C \ni c_1 = d_1 \vee E & \xrightarrow{F} & \\ & & c' \in C. \\ C \ni c_2 = d_2 \vee E & \xrightarrow{F} & \end{array}$$

Now we state our theorem.

**Theorem.** The following are equivalent:

(I)  $F \neq$  injective.

(I<sub>n</sub>)  $\exists d_1, d_2 \in C_{n-4}$  :  $n$ - mutually erasable for some  $n$  in  $\mathbb{N}$ .

(I'<sub>n</sub>)  $\exists d_1, d_2 \in C_{n-4}$  :  $n$ - not distinguished for some  $n$  in  $\mathbb{N}$ .

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## 3. PROOF FOR THE THEOREM

(a)

(I) :  $F \neq \text{injective} \Rightarrow \exists c_1, c_2 \in C$  such that(i)  $c_1 \neq c_2$ (ii)  $F(c_1) = F(c_2)$  $\Rightarrow$  Since $c_1, c_2$  : finite type,

we have

 $\exists d_1, d_2 \in C_{n-4}$  and $\exists E \in C \setminus C_{n-4}$  of which states are all  $q$  such that $c_i = d_i \vee E$  for  $i = 1, 2$ ,

where

(i)  $d_1 \neq d_2$  by (i),(ii)  $F(c_1) = F(c_2)$  by (ii) $\Rightarrow (I'_n)$ 

(b)

 $(I'_n) : \exists d_1, d_2 \in C_{n-4}$  with  $d_1 \neq d_2$  and  $\exists E \in C \setminus C_{n-4}$ 

such that

$$F(d_1 \vee E) = F(d_2 \vee E)$$

 $\Rightarrow$  expressing  $E$  as

$$E = g \vee h \vee E', \text{ where}$$

 $g$  in  $C_{n-2} \setminus C_{n-4}$ ,  $h$  in  $C_n \setminus C_{n-2}$  and $E'$  an environment of  $C_n$ ,we have  $d'$  in  $C_{n-4}$  and  $g'$  in  $C_{n-2} \setminus C_{n-4}$  such that

$$\begin{array}{ccc} d_1 \vee g & \xrightarrow{F_{n-2,h}} & \\ & \searrow & \\ & & d' \vee g', \\ & \nearrow & \\ d_2 \vee g & \xrightarrow{F_{n-2,h}} & \end{array}$$

 $\Rightarrow (I_n)$ .

(c)  
 $(I_n) : \exists d_1, d_2, d' \text{ in } C_{n-4} \text{ with } d_1 \neq d_2, \exists g, g' \text{ in } C_{n-2} \setminus C_{n-4},$   
 $\exists h \text{ in } C_n \setminus C_{n-2}$   
 such that

$$\begin{array}{ccc} d_1 \vee g & \xrightarrow{F_{n-2,h}} & d' \vee g' \\ d_2 \vee g & \xrightarrow{F_{n-2,h}} & \end{array}$$

$\Rightarrow$  by (b) of Remark

$\exists g_1 \text{ in } C_n \setminus C_{n-2}, \exists g_2 \text{ in } C_{n+2} \setminus C_n, \dots$  and  
 $\exists c \in C$

such that

$$\begin{array}{ccc} C \ni c_1 = d_1 \vee g \vee g_1 \vee g_2 \vee \dots & \xrightarrow{F} & \\ C \ni c_2 = d_2 \vee g \vee g_1 \vee g_2 \vee \dots & \xrightarrow{F} & c', \end{array}$$

$\Rightarrow c_1 \neq c_2$  by  $d_1 \neq d_2$ , and  $F(c_1) = F(c_2)$

$\Rightarrow$  (I).

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## REFERENCES

- [1] B. Durand, Global Properties of Cellular automata, [E. Goles, S. Martinez, Eds. , Cellular automata Complex Systems], Kluwer Academic Publishers, (1999) 1-22.
- [2] K. Jarkko, [Part 1: Spcial issue: Current research trends in cellular automata theory], Nat. Comput. 16, no. 3 (2017) 365-366.
- [3] E. F. Moore, Machine Models of Self-Reproduction, Proc. Symp. Apl. Math. 14 (1963) 17-34.
- [4] J. Myhill, The Convers to Moore's Garden-of-Eden Theorem, Proc. Am. Math. Soc. 14 (1963) 685-686.
- [5] H. Nishio, T. Saito, Information of Cellular Automata I - An Algebraic Sutady -, Fundamenta Informaticae 58 (2003) 399-420.

CHUOU 2-3-8, MOROYAMA, IRUMA, SAITAMA, JAPAN

*E-mail address:* `hishi@yuzu-tv.ne.jp`